Variational Approach to the Lifting Surface Problem

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Application of a variational principle to the nonself-adjoint lifting surface integral equation, first proposed by Flax, is given a closer look from the finite-element viewpoint. A scheme to avoid the usual introduction of the reverse flow is proposed. The plan form is divided into elements, and the unknown pressure loading is expressed as the sum of shape functions, each of which gives a local approximation within an element but vanishes elsewhere. The quadrature difficulty in the numerical solution can thus be effectively reduced. Because of the difference kernel in the integral equation, if both the plan form and the finite-element mesh pattern obey polar symmetry, the resulting algebraic problem is shown to have a symmetric coefficient matrix, in fact identical to that which follows from a straight Galerkin procedure using the same basis of approximating functions. For arbitrary plan form and mesh pattern, symmetrization can be achieved by an imbedding technique. Numerical examples are worked out for the classical two-dimensional flat plate and a number of lifting surfaces in steady incompressible flow, mainly to demonstrate the feasibility of the approach.

Introduction

THE case of the lifting surface is reopened here not with a challenge to current existing numerical programs, but as a testing ground to treat aerodynamic problems via the finite-element method. We note that, especially when only the Laplace equation is involved, the casting of a boundary value problem into a set of algebraic equations in the finite-element style is straightforward. There are excellent reviews, e.g., Argyris et al.[1] and Norrie and de Vries.[2] For two-dimensional airfoils in steady flow sample results of the calculation can also be found in Ref. 1. Some details of the method in practice are discussed by Shen.[3] Now, one of the difficulties of most aerodynamic applications is the necessity of simulating the infinite domain. In three-dimensional flows such as the lifting surface. it becomes crucial since three-dimensional finite elements are at best awkward, and a moderately large number of them will quickly tax the capacity of the available computer.

For this reason, we choose in this paper to deal with the integral equation formulation. The unknowns are now confined within the plan form, which is a two-dimensional region of finite size. Such a region is ideal for discretization into finite elements. The tradeoff here is that the resulting system matrix can no longer be sparse but must be a full one. Nevertheless, it is felt that this disadvantage should be relatively minor in comparison with the savings from the reduction in space dimensions.

Numerical solution of the integral equation has been extensively reviewed by Landahl and Stark.[4] The highly developed discrete lattice treatment satisfying the downwash requirement through collocation at reference points may be already regarded as a finite-element technique in a broad sense. For the two-dimensional airfoil, Argyris and Scharpf[5] looked at the integral equation from the finite-element viewpoint. They discussed the error control by collocation and by least square, and suggested the merits of using a better shape-function in each segment. Curiously enough no mention was made on the possibility of improving the accuracy through a variational principle, which has played a central role in most applications of the finite-element method in structural problems. To be sure, the lifting surface integral equation involves a nonself-adjoint operator, but a well known variational principle does exist. It has been adroitly exploited by Flax,[6] who obtained a set of "reverse flow theorems," including the effects of compressibility and oscillatory motion. He also indicated the possibility of using it to calculate the lift distribution. Attempts in the latter direction have since been carried out by Davies[7] and Stark.[8] In both cases, a lifting surface of the same plan form moving in the reverse direction was taken as the adjoint problem, and the two unknown lift distributions were represented by superposing suitable continuous "loading functions" in both the chordwise and spanwise directions. As a result, the numerical quadrature necessary in the variational principle appear to require careful handling.

We try to pursue the variational approach along a slightly different path. The central theme is the elimination of the separate reverse flow together with its approximate representation by loading functions. It is easy to see that under certain symmetry of the layout, the desired flow itself can serve as a convenient reverse flow. We further adopt the discretization as in standard finite-element procedures. The approximate lift distribution is now constructed from a base consisting of simple "shape functions" of compact support. The quadrature difficulty is thus largely reduced. Further, the resulting algebraic problem is shown to have a symmetric system matrix. For arbitrary planform and given mesh pattern and size of the finite elements, a symmetrization is required and may be achieved by adding a membrane, similar in spirit to the Evvard technique in treating the tip regions of a supersonic wing. Illustrative numerical examples confirm the feasibility and provide results that seem quite encouraging.

The Variational Principle and the Adjoint Problem

To facilitate further discussion we start with a brief summary. The lifting surface problem in aerodynamics may be formulated into a singular integral equation of the type

$$w(\mathbf{x}) = \int_S K(\mathbf{x}-\xi)\gamma(\xi)\, d\xi \tag{1}$$

where $w(\mathbf{x})$ is the downwash at point $\mathbf{x}$ within the planar region S representing the lifting surface, $\gamma(\xi)$ is the pressure loading at the point ξ in S, $K(\mathbf{x}-\xi)$ is the kernel function whose argument depends explicitly on the vector difference $\mathbf{x}-\xi$, and $d\xi$ is the area element around the point ξ in S. The approximation involved in the derivation of Eq. (1) as well as the explicit expressions

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of the kernel function K in various cases need not be repeated here. In operator form, let us write Eq. (1) as

$$w = L\gamma \quad (2)$$

where L stands for the preceding linear integral operator. Because $K(\mathbf{x}, \xi)$ is nonsymmetric with respect to $\mathbf{x}$ and ξ, i.e.,

$$K(\mathbf{x}, \xi) \neq K(\xi, \mathbf{x})$$

the operator L is nonself-adjoint. Solution for the loading γ in the most straightforward manner is usually by discretizing γ or expending γ in terms of "loading functions" in both chord and span-wise directions, and then collocating the downwash w at a number of reference points.

Because of the nonself-adjointness of L, let an adjoint operator L^* be defined so that

$$(v, Lu) = (u, L^*v)$$

where (a, b) is the inner product of two arbitrary functions $a(\mathbf{x})$ and $b(\mathbf{x})$ over S

$$(a, b) \equiv \int_s a(\mathbf{x})b(\mathbf{x})\, d\mathbf{x} \quad (3)$$

For Eq. (1), L^* is easy to identify and an adjoint problem is formulated as

$$w^*(\mathbf{x}) = L^*\gamma^* = \int_s K(\xi - \mathbf{x})\gamma^*(\xi)\, d\xi \quad (4)$$

If Eq. (4) is interpreted as a lifting surface problem of the same plan form S, then $\gamma^*(\xi)$ is its loading and $w^*(\mathbf{x})$ its downwash, noting, however, that the wing may not be a real one since $\mathbf{x}$ and ξ occupy wrong places in the kernel function. Consider next the functional I of the exact solutions γ and γ^*

$$I = (\gamma^*, w) + (\gamma, w^*) - (\gamma^*, L\gamma) \quad (5)$$

where terms like (γ^*, w) are again defined as in Eq. (3). By asuming w and w^* given and taking the first variation of I, it follows

$$\delta I = (\delta\gamma^*, w - L\gamma) + (\delta\gamma, w^* - L^*\gamma^*) = 0 \quad (6)$$

for arbitrary variations $\delta\gamma^*$ and $\delta\gamma$. Hence $\delta I = 0$ is a variation principle, its Euler equations agreeing with Eqs. (2) and (4).

To construct approximate solutions of γ and γ^*, one natural way is to expand each in terms of its appropriate loading functions

$$\gamma(\mathbf{x}) = \sum_n \gamma_n \phi_n(\mathbf{x})$$
$$\gamma^*(\mathbf{x}) = \sum_n \gamma_n^* \phi_n^*(\mathbf{x}) \quad (7)$$

where γ_n and γ_n^* are undetermined coefficients and the sets $\phi_n(\mathbf{x})$ and $\phi_n^*(\mathbf{x})$ may be in general different. In lifting surface problems, because of the leading-edge singularity of the pressure loading, it becomes convenient to take γ^* as the loading on a wing of the same plan form but in reverse motion, Fig. 1. The Kutta condition at the trailing edges then assures that the functional I can be evaluated. This viewpoint was adopted by all previous authors[6-8] and provides the guidelines for selecting the set ϕ_n^*.

Substitution of Eq. (7) into Eq. (6) yields for arbitrary $\delta\gamma_n$ and $\delta\gamma_n^*$

$$\sum_n (\phi_m^*, L\phi_n)\gamma_n = (\phi_m^*, w)$$
$$\sum_n (\phi_m, L^*\phi_n^*)\gamma_n^* = (\phi_m, w^*) \quad (8)$$

where m and n run through all the indices. It may be observed that in Eq. (8) the solutions γ_n and γ_n^* are completely uncoupled. The desired solution γ_n for the given lifting surface is obtained regardless of the specific downwash $w^*(\mathbf{x})$ chosen for the reverse flow problem. We can in fact consider the procedure for γ_n as identical to the Galerkin method applied to Eq. (1), except that the weight function belongs to the expansion of the reverse flow problem. The functional I, meanwhile, has a physical interpretation and its value depends on the choice of $w^*(\mathbf{x})$: when γ and γ^* are exact solutions, Eq. (5) is reduced to

$$I_{\text{exact}} = (\gamma^*, w) = (\gamma, w^*) \quad (9)$$

For example, if $w^* = 1$, I_{exact} gives the total lift of the given

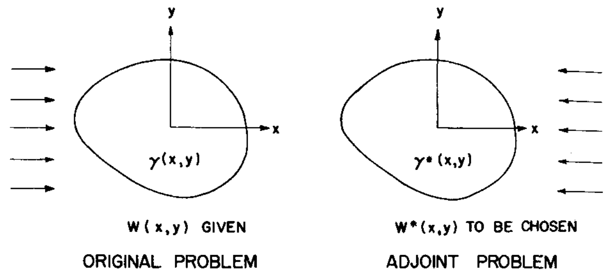

Fig. 1 Wing in reverse flow as adjoint problem.

wing, etc. The reciprocity relation Eq. (9) is fundamental to the reverse flow theorems obtainable from different choices of w^*. In addition, although Eq. (8) shows that the γ_n's are obtained independent of w^* directly, the choice of ϕ_m^* would affect the capability of representing the reverse flow with an assigned w^*. The accuracy will be reflected in the prediction of (γ, w^*).

Analysis for the Symmetrical Plan Form

As the variational principle requires an adjoint problem that is to some extent at our disposal, it should be of interest to study more closely what effects the degree of freedom w^* might have on the approximate solution constructed according to Eq. (6). For simplicity sake, let us restrict ourselves first to plan forms $S(\mathbf{x})$ said to have "polar symmetry," such that

$$S(-\mathbf{x}) = S(\mathbf{x}) \quad (10)$$

namely, all plan forms which may be mapped back to the original through reflections about both the x and the y axes. These include obviously the rectangular, and the elliptic with zero sweep of the midchord line, either in unyawed or yawed forward motion—the x-axis being the direction of the uniform flow. Some examples are shown in Fig. 2. The delta wing, however, is an exception.

With Eq. (10), it is almost intuitively clear that the adjoint reverse flow problem, Eq. (4), becomes identical to that of the original wing in forward flow but with a different downwash specification in Eq. (1). We prove this formally by substitution. Let us ask for an adjoint problem such that

$$\gamma^*(\mathbf{x}) = \gamma_1(-\mathbf{x}) \quad (11)$$

then the required $w^*(\mathbf{x})$ is, satisfying Eq. (4),

$$w^*(\mathbf{x}) = \int_{S(\xi)} K(\xi - \mathbf{x})\gamma_1(-\xi)\, d\xi$$

A simple change of variable $\xi' = -\xi$ leads to

$$w^*(\mathbf{x}) = \int_{S(\xi')} K(-\mathbf{x} - \xi')\gamma_1(\xi')\, d\xi' = w_1(-\mathbf{x}) \quad (12)$$

where $w_1(\mathbf{x})$ is the downwash required for the original wing in forward flow, Eq. (1), if the lift distribution $\gamma_1(\mathbf{x})$ is prescribed. Conversely, should $w^*(\mathbf{x})$ be chosen first for the reverse flow, γ^* is given by the solution γ_1 for the wing in forward flow with a downwash specified by

$$w_1(\mathbf{x}) = w^*(-\mathbf{x})$$

This result suggests that if the set $\phi_n(\mathbf{x})$ in Eq. (6) suffices for all forward flow problems of arbitrary $w(\mathbf{x})$, the set $\phi_n^*(\mathbf{x})$ for the reverse flow should be no other than

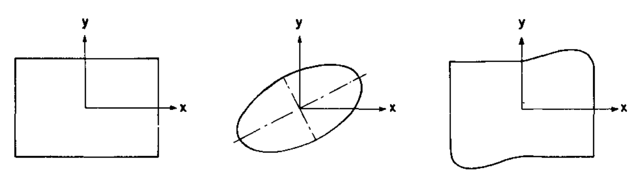

Fig. 2 Examples of plan forms with polar symmetry.

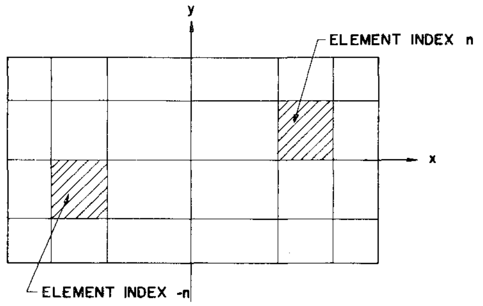

Fig. 3 Indexing of elements in symmetrical layout.

$$\phi_n^*(\mathbf{x}) = \phi_n(-\mathbf{x}) \tag{13}$$

provided the plan form obeys Eq. (10). Once $\phi_n^*(\mathbf{x})$ is chosen, the specific $w^*(\mathbf{x})$ is of no concern in the determination of the coefficients γ_n as already mentioned before.

The solution of γ_n, according to the first of Eq. (8), is now to be from the algebraic equations

$$\sum_n s_{mn}\gamma_n = w_m \tag{14}$$

where

$$\begin{aligned} s_{mn} &= (\phi_m(-\mathbf{x}), L\phi_n) \\ w_m &= (\phi_m(-\mathbf{x}), w(\mathbf{x})) \end{aligned} \tag{15}$$

Explicitly, for the difference kernel of Eq. (1),

$$\begin{aligned} s_{mn} &= \int_{S(\mathbf{x})}\int_{S(\xi)} \phi_m(-\mathbf{x})\phi_n(\xi)K(\mathbf{x}-\xi)\,d\xi\,d\mathbf{x} \\ &= \int_{S(\mathbf{x})}\int_{S(\xi)} \phi_m(\mathbf{x})\phi_n(\xi)K(-\mathbf{x}-\xi)\,d\xi\,d\mathbf{x} \end{aligned}$$

because $S(-\mathbf{x}) = S(\mathbf{x})$, Eq. (10). Interchanging next the variables $\mathbf{x}$ and ξ in the integral, we observe

$$s_{mn} = s_{nm} \tag{16}$$

The system matrix s_{mn} is seen to be symmetric, although the operator L is not.

These results, Eqs. (14–16), can be alternatively derived from the following: given $w(\mathbf{x})$, by taking $w^*(\mathbf{x}) = w(-\mathbf{x})$ the adjoint problem is made identical physically to the original one and, consequently, $\gamma^*(\mathbf{x}) = \gamma(-\mathbf{x})$. The functional I then depends on only one set of parameters γ_n, and first variations of γ_n are taken. This procedure automatically introduces Eq. (13), and brings some physical feeling into the picture.

Finite-Element Discretization

Importance of a Symmetrical Layout

We now subdivide the domain S into small pieces, each of which becomes a "finite element," and make simple local approximations of the unknown γ in each element. It is not our intention here to enter into an elaborate discussion of the finite-element procedure. For the general features, it is convenient to examine a special case. If the lifting distribution is approximated by a "constant loading" in each finite element, in Eq. (7) the coefficient γ_n may be identified as the local average value of γ in the element corresponding to the index n and $\phi_n(\mathbf{x})$ becomes the "shape function," having the property

$$\begin{aligned} \phi_n(\mathbf{x}) &= 1, \text{ for } \mathbf{x} \text{ within the element of index } n \\ &= 0, \text{ elsewhere} \end{aligned}$$

For the plan form let us consider a rectangular wing divided into rectangular finite elements. Note that the definition of the system matrix s_{mn}, Eq. (15), calls for the integration involving $\phi_n(-\mathbf{x})$, which has the value unity only over an area in polar symmetry with the element of index n. Thus for machine computation, it becomes highly desirable to lay out the finite elements so that this area in question coincides with one of the elements. In other words, we propose to lay out the finite elements in a pattern which maintains polar symmetry, too. Such is illustrated in Fig. 3. By suitable indexing, for a total of $2N$ elements covering the entire S, the index n runs from $-N$ to N, and the shape function obeys the relation

$$\phi_n(-\mathbf{x}) = \phi_{-n}(\mathbf{x}) \tag{17}$$

Evidently, this discussion is in no way restricted to the rectangular nature of either the plan form or the pattern. Indeed no assumption of ϕ_n is really needed. The constant loading is introduced only to help motivate the symmetrical layout.

Equivalence with Galerkin's Procedure

As a further consequence, so long as both the plan form and the finite element pattern have polar symmetry, it now can be shown that the algebraic system for γ_n, Eq. (14), derived from the variational principle, is in fact no different from what would result from a straight Galerkin procedure based upon the set ϕ_n satisfying Eq. (17).

In the Galerkin procedure, if Eq. (7) is written as the approximation, instead of Eq. (14) and (15), there will be

$$\sum_n s'_{mn}\gamma_n = w_m' \tag{18}$$

with

$$\begin{aligned} s'_{mn} &= [\phi_m(\mathbf{x}), L\phi_n] \\ w_m' &= [\phi_m(\mathbf{x}), w] \end{aligned} \tag{19}$$

Again, for the difference kernel of Eq. (1),

$$\begin{aligned} s'_{mn} &= \int_{S(\mathbf{x})}\int_{S(\xi)} \phi_m(\mathbf{x})\phi_n(\xi)K(\mathbf{x}-\xi)\,d\xi\,d\mathbf{x} \\ &= \int_{S(\mathbf{x})}\int_{S(\xi)} \phi_m(-\mathbf{x})\phi_n(\xi)K(-\mathbf{x}-\xi)\,d\xi\,d\mathbf{x} \end{aligned}$$

by reversing the sign of $\mathbf{x}$ and noting Eq. (10). Because of Eq. (17), it follows

$$\begin{aligned} s'_{mn} &= \int_{S(\mathbf{x})}\int_{S(\xi)} \phi_{-m}(\mathbf{x})\phi_n(\xi)K(-\xi-\mathbf{x})\,d\xi\,d\mathbf{x} \\ &= s_{-m,n} \end{aligned}$$

Likewise by comparing the definitions of w_m and w_m' in Eq. (15) and (19), respectively

$$w_m' = w_{-m}$$

Thus Eq. (18) can be written instead as

$$\sum_n s_{-m,n}\gamma_n = w_{-m} \tag{20}$$

With the index m running from $-N$ to N, Eqs. (20) and (14) are identical. Our claim is thus proved.

Symmetrization of Plan Form and the Membrane Concept

It is of interest to extend the preceding analysis to plan forms where Eq. (10) does not apply, so as to cover such cases as the delta wing, wings with sweep back and even multiple lifting surfaces. Furthermore, we may also wish to be free to adopt an irregular mesh pattern, putting more elements, for example, near the leading edge, without having to employ their counterparts near the trailing edge to maintain the polar symmetry of the layout. In other words, given an arbitrary plan form S and an arbitrary pattern of the finite elements on S, the question is how to devise a symmetrization scheme.

Our proposal is simply to imbed the surface S, with its mesh pattern, in a larger surface S' that has the required symmetry. To keep the physical problems identical, on S itself (now part of S') the downwash $w(\mathbf{x})$ is prescribed and $\gamma(\mathbf{x})$ unknown. On $S'-S$, which is the portion of S' outside of S, we require $\gamma = 0$ and accept $w(\mathbf{x})$ as that which is produced by the lift distribution on S. It amounts to consider $S'-S$ as an extension of the surface S, so deformed as to result in zero pressure difference across it—like a membrane. This membrane concept, of course, reminds one of the well-known Evvard technique in supersonic wing theory for treating the tip regions. It is important to remember, however, that the variation principle, Eq. (6), must be applied to the extended surface S' as if $w(\mathbf{x})$ was known everywhere on S' and $\gamma(\mathbf{x})$ to be determined also over all S'. The

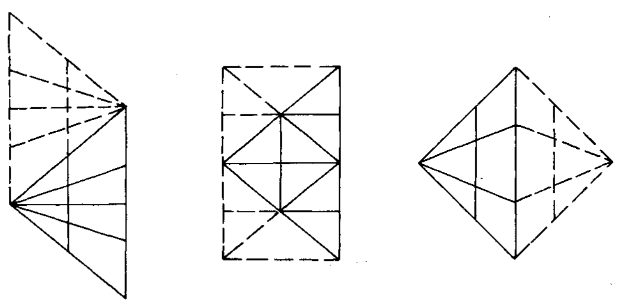

Fig. 4 Three possible ways to symmetrize a delta wing.

variation of γ must be first carried out over all S'. Then we set $\gamma = 0$ on $S' - S$ in the system Eq. (14). Equivalently, in the Galerkin procedure Eq. (20), the entire matrix s'_{mn} for all m, n in S' must be formed, and then set $\gamma_n = 0$ for the indices n covering $S' - S$. Figure 4 illustrates three different ways to symmetrize a delta wing. The wing and its element layout are in solid lines; the membrane and its elements are in broken lines. There are clearly many other possibilities. The proper choice is to be made from the additional consideration of the desired mesh pattern on S, which is to be kept symmetrical for the entire area S'.

Illustrative Numerical Examples

As feasibility tests of the preceding considerations put in practice, we have carried out computations for the classical two-dimensional airfoil and also a few lifting surfaces, all in steady incompressible flow. Thus, for the two-dimensional airfoil lying between $x = \pm c/2$, c being the chord,

$$w(x) = \frac{1}{2\pi} \int_{-c/2}^{c/2} \frac{\gamma(\xi)\, d\xi}{\xi - x}$$

for the lifting surface of plan form $s(x, y)$,

$$w(x, y) = \frac{1}{4\pi} \iint_s \frac{\gamma(\xi, \eta)}{(y-\eta)^2} \left[1 + \frac{x - \xi}{[(x-\xi)^2 + (y-\eta)^2]^{1/2}} \right] d\xi\, d\eta$$

The forms of $K(\xi - \mathbf{x})$ can be identified by comparison with Eq. (2).

The two-dimensional airfoil is particularly simple since the finite elements are only line segments. Three types of shape functions have been considered: 1) lumped loading—corresponding to using a delta function as $\phi_n(\mathbf{x})$, centered at a convenient point of application within the element; 2) constant loading—with $\phi_n(\mathbf{x})$ equal to unity within the element and zero outside; 3) linear loading—with $\phi_n(\mathbf{x})$ assuming the value unity at the "nodal point" $\mathbf{x} = \mathbf{x}_n$, where $\gamma = \gamma_n$, dropping to zero linearly in the two adjacent elements sharing the nodal point, and equal to zero elsewhere. In addition, to represent better the leading-edge singularity, we also experimented with a special shape function for the element at the leading edge, reproducing the square root singularity

$$\phi_0(\mathbf{x}) = 1/(x)^{1/2}, \qquad 0 < x < x_1$$
$$= 0$$

elsewhere, if the leading edge is at $x = 0$, and the element between $x = 0$ and $x = x_1$. These shape functions are shown in Fig. 5. Similar special shape functions can be introduced in other regions of singular behavior, such as due to control surface deflection.

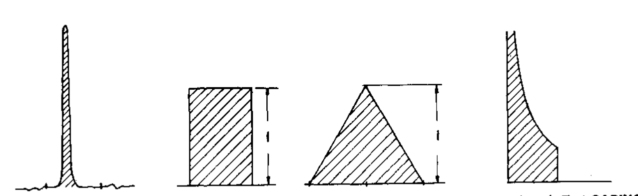

Fig. 5 Examples of shape functions.

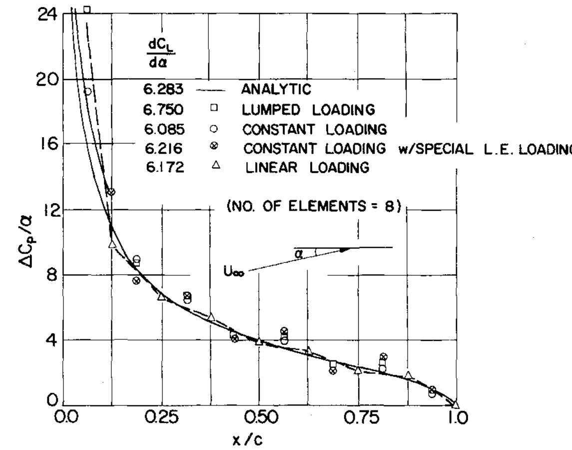

Fig. 6 Comparison of analytic and predicted chordwise loading for a flat plate by various approximate schemes.

The lumped loading, of course, is no other than the usual approximation of concentrated vortices. Strict application of it as $\phi_n(\mathbf{x})$ leads to directly the collocation scheme, with the control point for downwash right at the vortex itself. In the popular vortex lattice theory, e.g., Giesing,[9] it is advocated to put the vortex at the $\frac{1}{4}$-point of the element and the collocation point for the downwash at the $\frac{3}{4}$-point, both distances being measured from the front boundary (nearer to the leading edge) of the element. This model cannot be rigorously accommodated by our simple shape functions. However, it may be regarded as a quadrature formula in the evaluation of the system matrix s_{mn}.

Another detail is the trailing edge Kutta condition. If the trailing edge is a nodal point, we simply set the nodal value $\gamma_T = 0$ and allow no variation of γ_T. In cases of lumped and constant loadings, we assume the loading $\gamma(\mathbf{x})$ to drop to zero linearly in the segment containing the last two elements next to the trailing edge. The unknown γ_n in the last element before the trailing edge then is assigned by interpolation a value in terms of the one pertaining to the element ahead of it.

The flat plate at angle of attack was fairly exhaustively studied to gain experience. Effects of shape function and approximate quadrature are shown in Fig. 6, with the chord divided into equal length elements. (Calculated results for lumped loading

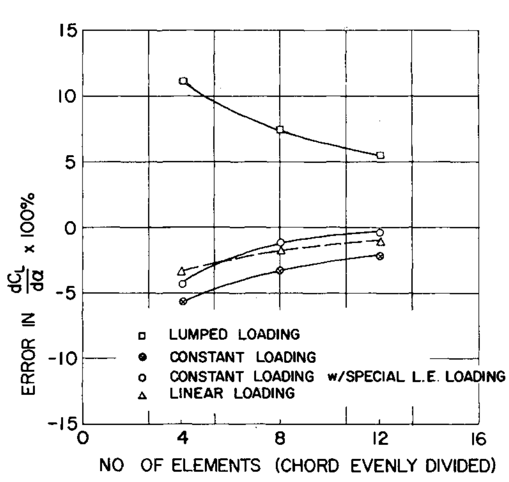

Fig. 7 Convergence studies of lift coefficient for a flat plate by various approximation schemes.

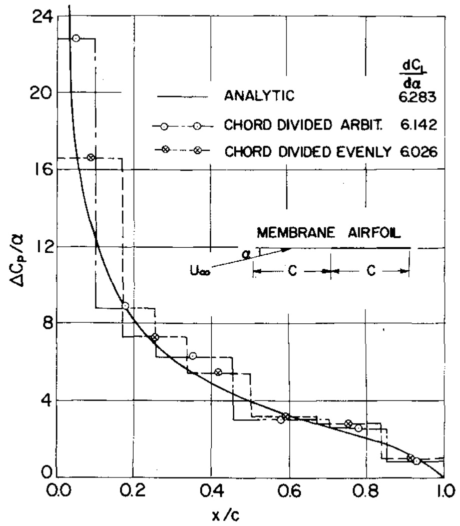

Fig. 8 Comparison of analytic and predicted chordwise loading for a flat plate with different divisions for the chord.

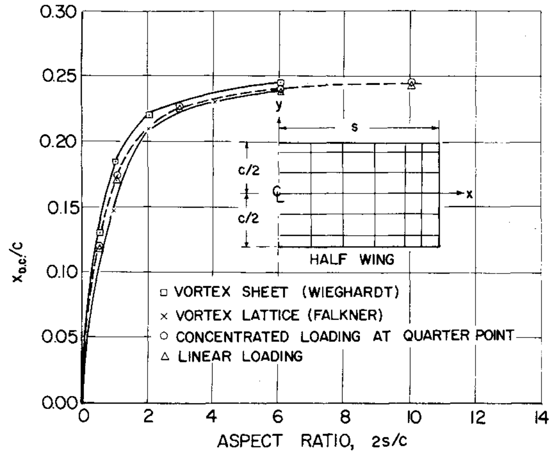

Fig. 10 Comparison of predicted aerodynamic center for rectangular wings by different methods.

have been smeared to yield an average within each element.) A somewhat unexpected feature is that the values of γ_n in adjacent elements appear actually to scatter alternately above and below the exact solution. This behavior presumably reflects the nonself-adjointness of the operator L. It is also noted that the average of any two adjacent points turns out to be much closer to the exact solution evaluated at the midpoint between them. In Fig. 7 is shown the convergence of the lift curve slope as a function of the number of elements, for the various shape functions. Improvements of accuracy from applying the membrane concept to flat plate divided into unequal elements is demonstrated in Fig. 8.

For lifting surface calculations, the mesh pattern must first be decided. We have used only rectangular elements symmetrically laid out over the plan form S (or extended to S by membrane if necessary). As shape functions the following two types are employed: 1) lumped loading, in the form of a lifting line in each element placed along the front $\frac{1}{4}$-point from the element leading edge, thus directly comparable to the vortex lattices in Giesing's method[9]; 2) linear loading in the chordwise direction but constant in the spanwise direction.

To reduce the numerical work, the control point concept is partially retained. For further quadrature involving the induced downwash, the variation of the latter in each element is neglected and its constant value is assumed to be approximated by that evaluated at the control point. In the lumped loading case, the control point of an element is chosen at the $\frac{3}{4}$-point in the chordwise direction and midway between the spanwise boundaries of the element, as commonly done in lattice methods. In the linear loading case, it is taken to be the centroid of the element. Actual evaluation of the induced downwash at a control point is carried out by a 3-point gaussian formula (in both chordwise and spanwise directions), the singular behavior having been first removed by the usual partial integration. Our numerical studies cover a number of lifting surfaces of the rectangular plan form, the circular wing and a delta wing.

For each rectangular wing, the finite elements are symmetrically laid out as in Fig. 10. A total of 36 elements are used to represent half the wing. Calculated variations of the lift curve slope and the aerodynamic center with aspect ratio are shown in Figs. 9 and 10, respectively. Our results are in close agreement with those due to Wieghardt[10] and Falkner.[11] The two shape functions apparently yield nearly identical answers in these over-all characteristics.

For the circular wing, 60 rectangular elements on each half of the wing are used and arranged to approximate the boundary geometry. The layout (symmetrical fore-and-aft) and the results for the aerodynamic center locus can be seen in Fig. 11. Again, the aerodynamic center appears to be rather insensitive to the method of calculation. We, therefore, turn to a comparison of the pressure distribution. A typical case, along the cord at $y/s = 0.5$, is shown as Fig. 12. Here the ambiguity in converting the lumped loading distribution to pressure coefficients should be noted. The practice in lattice methods is to divide the total lift in each element by the element area, and assume the result to represent the pressure loading at the center of the local lifting

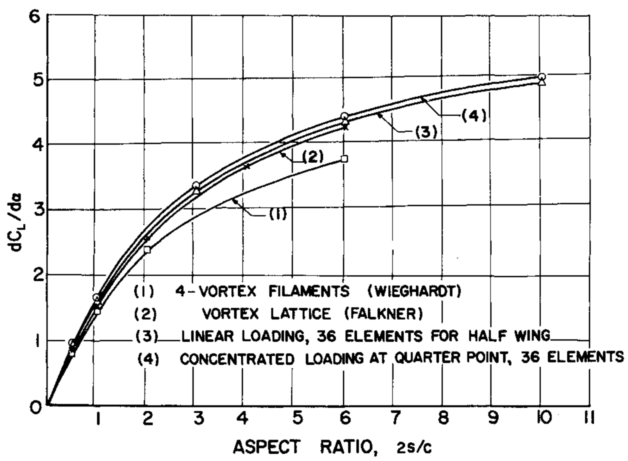

Fig. 9 Comparison of predicted lift coefficient for rectangular wings by different methods.

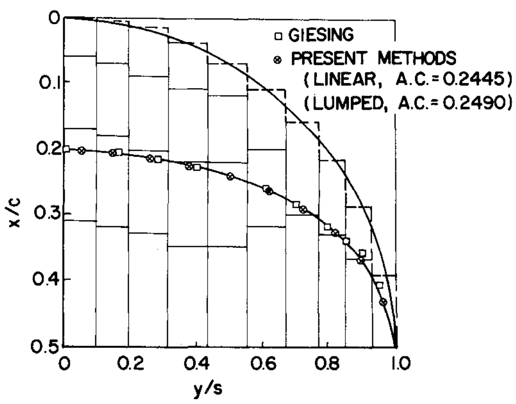

Fig. 11 Circular wing—finite element representation in a quadrant and comparison of predicted aerodynamic center.

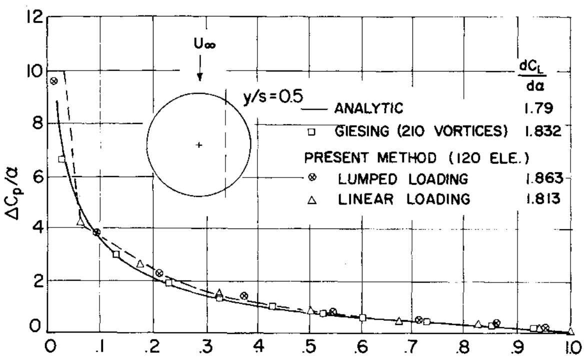

Fig. 12 Comparison of analytic and predicted chordwise loading for a circular wing at $y/s = 0.5$.

line. Such a procedure, strictly speaking, is not self-consistent, since the pressure loading so obtained does not yield the correct aerodynamic center. On this basis, however, our lumped loading results may be compared against those from Giesing. They seem to follow the same curve, but seriously only the prediction of our linear loading is to be compared with that from analytic solution. The linear loading results deviate from the analytic curve mainly near the leading edge, as expected. The desirability of using a special element reflecting the leading-edge singular behavior is clearly suggested. For the total lift, the linear loading is seen to be superior to the lattice method using many more unknowns.

For a delta wing of aspect ratio 4, we first symmetrize the plan form by adding a membrane to the rear of the trailing edge to form a rhombus, which permits any element layout on the delta itself. On half of the delta 50 elements are used. The layout of the elements and the computed aerodynamic center locus are shown in Fig. 13, and the pressure distributions along two chordwise sections are presented in Fig. 14. Compared against Falkner's results, there is noticeable difference in the pressure distributions. Insofar as the lift curve slope is concerned, Falkner's prediction agrees with our lumped loading. Experiences above with both the flat plate and the circular wing suggest that the present procedure using linear loading should yield better accuracy than lumped loading.

Conclusions

We demonstrate above how it is possible to circumvent the introduction of a separate reverse flow in applying the variational method to the lifting surface integral equation. The required symmetrization is particularly easy in a discretization such as the finite-element procedure. That the resulting equations become identical to those from a direct application of Galerkin's procedure (to the symmetrized problem) and that the system matrix is thus made symmetric do not seem to have been noted before. These rather general observations should have implications in other problems governed by similar nonself-adjoint integral operators with a difference kernel. Our numerical examples are only illustrative. Whether in actual application to the lifting surface, the theoretical advantage of a variational method over the highly developed current schemes based on collocation can be realized, with acceptable penalty (in added complication), is a question that remains to be studied.

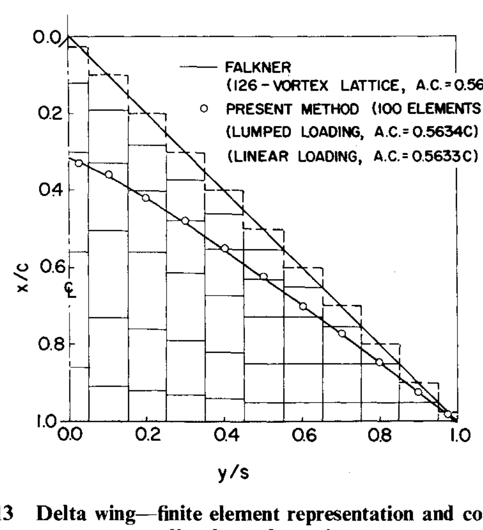

Fig. 13 Delta wing—finite element representation and comparison of predicted aerodynamic center.

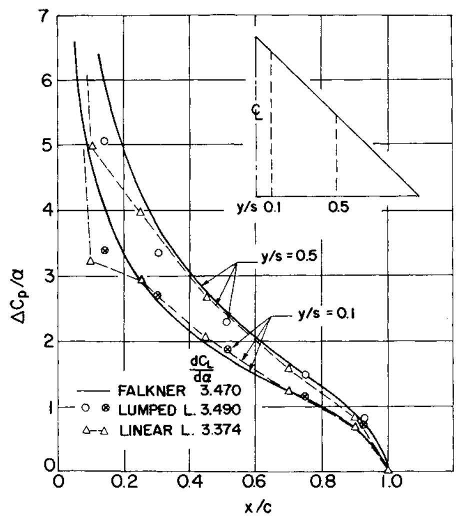

Fig. 14 Comparison of predicted chordwise loading at different sections of a delta wing.

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